

The quantum theory of scalar fields on the de Sitter expanding universe

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Abstract

New quantum modes of the free scalar field are derived in a special time-evolution picture that may be introduced in moving charts of de Sitter backgrounds. The wave functions of these new modes are solutions of the Klein-Gordon equation and energy eigenfunctions, defining the energy basis. This completes the scalar quantum mechanics where the momentum basis is well-known from long time. In this enlarged framework the quantization of the scalar field can be done in canonical way obtaining the principal conserved one-particle operators and the Green functions.

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1 Introduction

The quantum scalar field is the main piece in problems concerning quantum effects in the presence of gravitation or in quantum gravity and cosmology. Of a special interest in cosmology is the de Sitter (dS) expanding universe carrying scalar fields variously coupled with gravitation. In the quantum theory a special role is played by the free fields (minimally coupled with gravitation) since these are the principal ingredients in calculating scattering amplitudes using perturbations.

In curved spacetimes the field equations have covariant forms such that their operators are globally defined. If the manifold has, in addition, some isometries then the corresponding Killing vectors give rise to the generators of the representations of the isometry group carried by spaces of matter fields [1]. These generators are operators of the quantum mechanics whose main virtue is to commute with the operator of the field equation. On the other hand, these operators are related to the conserved quantities predicted by the Noether theorem from which one constructs the conserved one-particle operators of the quantum field theory. For this reason we say here that the isometry generators, or any other operator which commutes with that of the field equation, are *conserved* operators of the relativistic quantum mechanics. In general, the quantum modes can be globally defined on curved manifolds using field equations and suitable sets of commuting conserved operators. This method is useful in the case of the four-dimensional dS background which has $SO(4, 1)$ isometries.

The quantum modes of the scalar field in moving frames of dS manifolds are well-known from long time [2, 3]. Their wave functions are solutions of the Klein-Gordon (KG) equation and eigenfunctions of the conserved momentum operator but these are not able to provide us with information about energy. This is because in dS moving frames the operator $i\partial_t$ is not a Killing vector and, therefore, it is not conserved. We suggested [4, 5] that the correct energy operator must be the time-like Killing vector field of the dS geometry, as in the case of the anti-de Sitter manifolds [6, 7]. Moreover, we studied other conserved operators arising from dS isometries, showing that the energy operator does not commute with the momentum one [5, 8]. This means that the momentum and energy can not be measured simultaneously with desired accuracy. In other respects, it is known that there are no mass-shells. Obviously, these major difficulties may be avoided using new special methods for analyzing the time evolution of the free fields.

Recently we constructed a new Dirac quantum mechanics on spatially flat Robertson-Walker spacetimes in which we defined different time evolution pictures [8]. In the case of dS backgrounds these pictures helped us to find new solutions of the Dirac equation and to understand how can be measured the momentum and energy. We started with the *natural* picture (NP) which is just the usual Dirac theory in dS moving frames as it results from its Lagrangian. In this picture we found the complete system of fundamental spinors with well-determined momentum and helicity [5]. The NP can be transformed in a new time-evolution picture where the Dirac equation do not depend explicitly on

time. This is the Schrödinger picture (SP) we have introduced for studying the energy quantum modes of the Dirac field which can not be derived in other conjectures [8]. In SP we derived the complete set of fundamental solutions of the Dirac equation of given energy, momentum direction and helicity [9]. We must specify that our SP defined at the level of the relativistic quantum mechanics is completely different from the *functional* Schrödinger-picture of the quantum field theory [10] which was used for studying scalar fields [11].

Here we would like to continue our study using the same approach for improving the quantum theory of the massive charged scalar field in dS moving frames. We start with the NP where the time evolution is governed by the well-known KG equation which depends explicitly on time [2]. This produces quantum modes depending on momentum whose wave functions constitute a complete set of normalized fundamental solutions, defining the momentum basis. As in the Dirac case, we can define the SP where the KG equation does not depend explicitly on time. In this picture we derive the new complete set of normalized fundamental solutions of given energy and momentum direction which define the energy basis. Using these bases a coherent quantum mechanics can be constructed as a starting point to a suitable quantum theory of fields. Our objective here is to follow this way deriving step by step all the elements of the quantum theory of the scalar field in dS moving frames, up to the form of the Green functions related to the commutator ones.

We start in the second section with a short introduction in the scalar quantum mechanics on dS backgrounds defining the new SP. The next section is devoted to the fundamental solutions of the momentum and, respectively, energy bases. The wave functions of the energy basis are studied giving details since these represent the main new result obtained here. The transition coefficients between these bases are also written down in NP. In the fourth section we perform the quantization in canonical manner obtaining the form of the conserved one-particle operators in both the bases we use. Finally, we present the properties of the principal Green functions related to the commutator ones. The concluding remarks includes some observations on some specific features of the energy and momentum measurements on dS spacetimes.

2 Scalar quantum mechanics on dS spacetimes

Let us consider M be the dS spacetime, defined as the hyperboloid of radius $\frac{1}{\omega}$ in the five-dimensional flat spacetime M^5 of coordinates z^A ($A, B, \dots = 0, 1, 2, 3, 5$) and metric $\eta^5 = \text{diag}(1, -1, -1, -1, -1)$ [12]. The hyperboloid equation $\omega^2 \eta_{AB}^5 z^A z^B = -1$ defines M as the homogeneous space of the pseudo-orthogonal group $SO(4, 1)$ which is at the same time the gauge group of the metric η^5 and the isometry group, $I(M) \equiv SO(4, 1)$, of the dS spacetime. Then, it is convenient to use the covariant real parameters $\xi^{AB} = -\xi^{BA}$ since in this case the orbital basis-generators of the scalar representation of $SO(4, 1)$, carried by the space of the scalar fields over M^5 , have the standard form

$$L_{AB}^5 = i [\eta_{AC}^5 z^C \partial_B - \eta_{BC}^5 z^C \partial_A]. \quad (1)$$

They will give us directly the Killing vectors $k_{(AB)}$ of M which define the basis-generators

$$L_{(AB)} = -ik_{(AB)}^\mu \partial_\mu \quad (2)$$

of the scalar representation of $I(M)$. These operators can be calculated in any local chart $\{x\}$ of coordinates x^μ ($\mu, \nu, \dots = 0, 1, 2, 3$) where we know the functions $z^A(x)$.

In an arbitrary chart $\{x\}$ the action of a charged scalar field ϕ of mass m , minimally coupled with the gravitational field, reads

$$\mathcal{S}[\phi, \phi^*] = \int d^4x \sqrt{g} \mathcal{L} = \int d^4x \sqrt{g} (\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi), \quad (3)$$

where $g = |\det(g_{\mu\nu})|$. This action gives rise to the KG equation

$$\frac{1}{\sqrt{g}} \partial_\mu [\sqrt{g} g^{\mu\nu} \partial_\nu \phi] + m^2 \phi = 0. \quad (4)$$

The conserved quantities predicted by the Noether theorem can be calculated with the help of the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \phi^* \partial_\nu \phi + \partial_\nu \phi^* \partial_\mu \phi - g_{\mu\nu} \mathcal{L}. \quad (5)$$

Thus, for each isometry corresponding to a Killing vector $k_{(AB)}$ there exists the conserved current $\Theta^\mu[k_{(AB)}] = -T_{\nu}^{\mu} k_{(AB)}^\nu$ which satisfies $\Theta^\mu[k_{(AB)}]_{;\mu} = 0$ producing the conserved quantity [12]

$$C[k_{(AB)}] = \int_{\Sigma} d\sigma_\mu \sqrt{g} \Theta^\mu[k_{(AB)}], \quad (6)$$

on a given hypersurface $\Sigma \subset M$. Moreover, generalizing the form of the conserved electric charge due to the internal $U(1)$ symmetry one defines the relativistic scalar product of two scalar fields as [2]

$$\langle \phi, \phi' \rangle = i \int_{\Sigma} d\sigma^\mu \sqrt{g} \phi^* \overleftrightarrow{\partial}_\mu \phi', \quad (7)$$

using the notation $f \overleftrightarrow{\partial} h = f(\partial h) - h(\partial f)$. With this definition one obtains the following identities

$$C[k_{(AB)}] = \langle \phi, L_{(AB)} \phi \rangle \quad (8)$$

which can be proved for any Killing vector using the field equation (4) and the Green's theorem. These identities will be useful in quantization, giving directly the conserved one-particle operators of the quantum field theory.

As in the Dirac case [8], we say that the NP is the genuine quantum theory in the chart $\{t, \mathbf{x}\}$ with Cartesian coordinates and the Robertson-Walker line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = dt^2 - e^{2\omega t} (d\mathbf{x} \cdot d\mathbf{x}), \quad (9)$$

where $\sqrt{g} = e^{3\omega t}$. In this picture the time evolution of the massive scalar field is governed by the KG equation,

$$(\partial_t^2 - e^{-2\omega t} \Delta + 3\omega \partial_t + m^2) \phi(x) = 0. \quad (10)$$

The solutions of this equation may be square integrable functions or tempered distributions with respect to the scalar product (7) that in NP and for $\Sigma = \mathbb{R}^3$ takes the form

$$\langle \phi, \phi' \rangle = i \int d^3x e^{3\omega t} \phi^*(x) \overleftrightarrow{\partial}_t \phi'(x). \quad (11)$$

The principal operators of NP, the energy \hat{H} , momentum $\hat{\mathbf{P}}$ and coordinate $\hat{\mathbf{X}}$, have the same forms as in special relativity,

$$(\hat{H}\phi)(x) = i\partial_t \phi(x), \quad (\hat{P}^i \phi)(x) = -i\partial_i \phi(x), \quad (\hat{X}^i \phi)(x) = x^i \phi(x). \quad (12)$$

The operators \hat{X}^i and \hat{P}^i are time-independent and satisfy the well-known canonical commutation relations

$$[\hat{X}^i, \hat{P}^j] = i\delta_{ij}I, \quad [\hat{H}, \hat{X}^i] = [\hat{H}, \hat{P}^i] = 0, \quad (13)$$

where I is the identity operator.

Since the scalar theory on M has the high symmetry induced by the isometry group $I(M)$, the basis generators $L_{(AB)}$ are conserved operators commuting with the KG operator of Eq. (10). In this conjecture, we define the conserved energy operator [5, 8],

$$H \equiv \omega L_{(05)} = \hat{H} + \omega \hat{\mathbf{X}} \cdot \hat{\mathbf{P}}, \quad (14)$$

and verify that the momentum and angular momentum operators are also conserved since we have

$$\hat{P}^i \equiv \omega (L_{(5i)} - L_{(0i)}), \quad L_k \equiv \frac{1}{2} \varepsilon_{ijk} L_{(ij)} = \varepsilon_{ijk} \hat{X}^i \hat{P}^j. \quad (15)$$

In addition, there exists more three conserved generators,

$$R^i(t) \equiv L_{(5i)} + L_{(0i)} = \left(\frac{e^{-2\omega t}}{\omega} I - \omega \hat{\mathbf{X}}^2 \right) \hat{P}^i + 2\hat{X}^i H, \quad (16)$$

which do not have an immediate physical significance [5]. The $SO(4, 1)$ transformations corresponding to these basis-generators and the associated isometries are briefly presented in Ref. [5]. The conserved energy operator satisfy the commutation relations

$$[H, \hat{P}^i] = i\omega \hat{P}^i, \quad [H, \hat{X}^i] = -i\omega \hat{X}^i, \quad (17)$$

showing that the measurements of these observables are affected by uncertainty,

The NP can be changed using point-dependent operators which could be even non-unitary operators since the relativistic scalar product does not have a direct physical meaning as that of the non-relativistic quantum mechanics. We exploited this opportunity for defining the Schrödinger picture with the help of

the transformation $\phi(x) \rightarrow \phi_S(x) = U(x)\phi(x)$ produced by the operator of time dependent *dilatations* [8]

$$U(x) = \exp [-\omega t(x^i \partial_i)] . \quad (18)$$

This has the remarkable property

$$U^+(x) = e^{3\omega t} U^{-1}(x) , \quad (19)$$

and the following convenient action

$$U(x)F(x^i)U^{-1}(x) = F(e^{-\omega t}x^i) , \quad U(x)G(\partial_i)U^{-1}(x) = G(e^{\omega t}\partial_i) , \quad (20)$$

upon any analytical functions F and G . This transformation leads to the KG equation of the SP

$$\left[(\partial_t + \omega x^i \partial_i)^2 - \Delta + 3\omega(\partial_t + \omega x^i \partial_i) + m^2 \right] \phi_S(x) = 0 , \quad (21)$$

and allow us to define the scalar product of this picture,

$$\langle \phi_S, \phi'_S \rangle \equiv \langle \phi, \phi' \rangle = i \int d^3x \left[\phi_S^* \vec{\partial}_t \phi'_S + \omega x^i (\phi_S^* \vec{\partial}_i \phi'_S) \right] , \quad (22)$$

as it results from Eqs. (18) and (19).

The specific operators of SP, H_S , P_S^i and X_S^i , are defined as

$$(H_S \phi_S)(x) = i \partial_t \phi_S(x) , \quad (P_S^i \phi_S)(x) = -i \partial_i \phi_S(x) , \quad (X_S^i \phi_S)(x) = x^i \phi_S(x) , \quad (23)$$

obeying commutation relations similar to Eqs. (13). The meaning of these operators can be understood in the NP. Performing the inverse transformation we find that

$$U^{-1}(x) H_S U(x) = H , \quad (24)$$

and the new interesting time-dependent operators of NP,

$$X^i(t) = U^{-1}(x) X_S^i U(x) = e^{\omega t} \hat{X}^i , \quad (25)$$

$$P^i(t) = U^{-1}(x) P_S^i U(x) = e^{-\omega t} \hat{P}^i , \quad (26)$$

satisfying the usual commutation relations (13). The angular momentum has the same expression in both these pictures since it commutes with $U(x)$. We note that even if $X^i(t)$ and $P^i(t)$ commute with H they are not conserved operators since they do not commute with the KG operator.

In NP picture the eigenvalues problem $H f_E(t, \mathbf{x}) = E f_E(t, \mathbf{x})$ of the energy operator (24) leads to energy eigenfunctions of the form

$$f_E(t, \mathbf{x}) = F[e^{\omega t} \mathbf{x}] e^{-iEt} \quad (27)$$

where F is an arbitrary function. This explains why in this picture one can not find energy eigefunctions separating variables. However, in our SP these eigenfunctions become the new functions

$$f_E^S(t, \mathbf{x}) = U(x) f_E(t, \mathbf{x}) = F(\mathbf{x}) e^{-iEt} \quad (28)$$

which have separated variables. This means that in SP new quantum modes could be derived using the method of separating variables in coordinates or even in momentum representation.

3 Scalar plane waves

The specific feature of the quantum mechanics on M is that the energy operator (14) obeys Eqs. (17) which means that the conserved energy and momentum can not be measured simultaneously with desired accuracy. Consequently, there are no particular solutions of the KG equation with well-determined energy and momentum, being forced to consider different plane waves solutions depending either on momentum or on energy and momentum direction. Thus we shall work with two bases of fundamental solutions namely the momentum and energy ones.

3.1 The momentum basis

It is known that the KG equation (10) of NP can be analytically solved in terms of Bessel functions [2]. There are fundamental solutions determined as eigenfunctions of the set of commuting operators $\{\hat{P}^i\}$ of NP whose eigenvalues p^i are components of the momentum \mathbf{p} . Among different versions of solutions which are currently used we prefer the normalized solutions of positive frequencies that read

$$f_{\mathbf{p}}(x) = \frac{1}{2} \sqrt{\frac{\pi}{\omega}} \frac{1}{(2\pi)^{3/2}} e^{-3\omega t/2} Z_k \left(\frac{p}{\omega} e^{-\omega t} \right) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (29)$$

where the functions Z_k are defined in the Appendix A, $p = |\mathbf{p}|$ and we denote

$$k = \sqrt{\mu^2 - \frac{9}{4}}, \quad \mu = \frac{m}{\omega}, \quad (30)$$

provided $m > 3\omega/2$. Obviously, the fundamental solutions of negative frequencies are $f_{\mathbf{p}}^*(x)$. All these solutions satisfy the orthonormalization relations

$$\langle f_{\mathbf{p}}, f_{\mathbf{p}'} \rangle = -\langle f_{\mathbf{p}}^*, f_{\mathbf{p}'}^* \rangle = \delta^3(\mathbf{p} - \mathbf{p}'), \quad (31)$$

$$\langle f_{\mathbf{p}}, f_{\mathbf{p}'}^* \rangle = 0, \quad (32)$$

and the completeness condition

$$i \int d^3p f_{\mathbf{p}}^*(t, \mathbf{x}) \overleftrightarrow{\partial}_t f_{\mathbf{p}}(t, \mathbf{x}') = e^{-3\omega t} \delta^3(\mathbf{x} - \mathbf{x}'). \quad (33)$$

For this reason we say that the set $\{f_{\mathbf{p}} | \mathbf{p} \in \mathbb{R}_p^3\}$ forms the complete system of fundamental solutions of the *momentum* basis of NP. In this basis, the KG field can be expanded in terms of plane waves of positive and negative frequencies in usual manner as

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) = \int d^3p [f_{\mathbf{p}}(x)a(\mathbf{p}) + f_{\mathbf{p}}^*(x)b^*(\mathbf{p})] \quad (34)$$

where a and b are the wave functions of the momentum representation that can be calculated using the inversion formulas

$$a(\mathbf{p}) = \langle f_{\mathbf{p}}, \phi \rangle, \quad b(\mathbf{p}) = \langle f_{\mathbf{p}}, \phi^* \rangle. \quad (35)$$

3.2 The energy basis

The plane waves of given energy have to be derived in the SP where the KG equation has the suitable form (21). We assume that in this picture the KG field can be expanded as

$$\begin{aligned}\phi_S(x) &= \phi_S^{(+)}(x) + \phi_S^{(-)}(x) \\ &= \int_0^\infty dE \int d^3q \left[\hat{\phi}_S^{(+)}(E, \mathbf{q}) e^{-i(Et - \mathbf{q} \cdot \mathbf{x})} + \hat{\phi}_S^{(-)}(E, \mathbf{q}) e^{i(Et - \mathbf{q} \cdot \mathbf{x})} \right] \quad (36)\end{aligned}$$

where $\hat{\phi}_S^{(\pm)}$ behave as tempered distributions on the domain \mathbb{R}_q^3 such that the Green theorem may be used. Then we can replace the momentum operators P_S^i by q^i and the coordinate operators X_S^i by $i\partial_{q_i}$ obtaining the KG equation of the SP in momentum representation,

$$\begin{aligned}& \left\{ [\pm iE + \omega (q^i \partial_{q_i} + 3)]^2 \right. \\ & \left. - 3\omega [\pm iE + \omega (q^i \partial_{q_i} + 3)] + \mathbf{q}^2 + m^2 \right\} \hat{\phi}_S^{(\pm)}(E, \mathbf{q}) = 0, \quad (37)\end{aligned}$$

where E is the energy defined as the eigenvalue of H_S . We remind the reader that the operators P_S^i and X_S^i become in NP the time dependent operators (25) and respectively (26) while H_S is related through Eq. (24) to the conserved energy operator H . This means the energy E is a conserved quantity but the momentum \mathbf{q} does not have this property. More specific, only the scalar momentum $q = |\mathbf{q}|$ is not conserved while the momentum direction is conserved since the operator (26) is parallel with the conserved momentum $\hat{\mathbf{P}}$. For this reason we denote $\mathbf{q} = q \mathbf{n}$ observing that the differential operator of Eq. (37) is of radial type and reads $q^i \partial_{q_i} = q \partial_q$. Consequently, this operator acts only on the functions depending on q while the functions which depend on the momentum direction \mathbf{n} behave as constants. Therefore, we have to look for solutions of the form

$$\hat{\phi}_S^{(+)}(E, \mathbf{q}) = h_S(E, q) a(E, \mathbf{n}), \quad (38)$$

$$\hat{\phi}_S^{(-)}(E, \mathbf{q}) = [h_S(E, q)]^* b^*(E, \mathbf{n}), \quad (39)$$

where the function h_S satisfies an equation derived from Eq. (37) that can be written simply in the new variable $s = q/\omega$ and using the notations (30) and $\epsilon = E/\omega$. This equation,

$$\left[\frac{d^2}{ds^2} + \frac{2i\epsilon + 4}{s} \frac{d}{ds} + \frac{\mu^2 - \epsilon^2 + 3i\epsilon}{s^2} + 1 \right] h_S(\epsilon, s) = 0, \quad (40)$$

is of the Bessel type having solutions of the form $h_S(\epsilon, s) = \text{const } s^{-i\epsilon-3/2} Z_k(s)$. Collecting all the above results we derive the final expression of the KG field (36) as

$$\phi_S(x) = \int_0^\infty dE \int_{S^2} d\Omega_n \left\{ f_{E, \mathbf{n}}^S(x) a(E, \mathbf{n}) + [f_{E, \mathbf{n}}^S(x)]^* b^*(E, \mathbf{n}) \right\}, \quad (41)$$

where the integration covers the sphere $S^2 \subset \mathbb{R}_p^3$. The fundamental solutions $f_{E,\mathbf{n}}^S$ of positive frequencies, with energy E and momentum direction \mathbf{n} result to have the integral representation

$$f_{E,\mathbf{n}}^S(x) = N e^{-iEt} \int_0^\infty ds \sqrt{s} Z_k(s) e^{i\omega s \mathbf{n} \cdot \mathbf{x} - i\epsilon \ln s}, \quad (42)$$

where N is a normalization constant.

For understanding the physical meaning of this result we must turn back to NP where the scalar field

$$\phi(x) = \int_0^\infty dE \int_{S^2} d\Omega_n \{f_{E,\mathbf{n}}(x) a(E, \mathbf{n}) + [f_{E,\mathbf{n}}(x)]^* b^*(E, \mathbf{n})\}, \quad (43)$$

is expressed in terms of the solutions of NP which can be calculated as

$$f_{E,\mathbf{n}}(x) = U^{-1}(x) f_{E,\mathbf{n}}^S(x) = N e^{-iEt} \int_0^\infty ds \sqrt{s} Z_k(s) e^{i\omega s \mathbf{n} \cdot \mathbf{x}_t - i\epsilon \ln s}, \quad (44)$$

where $\mathbf{x}_t = e^{\omega t} \mathbf{x}$. Finally, changing the integration variable, $e^{\omega t} s \rightarrow s$, we obtain the definitive integral representation

$$f_{E,\mathbf{n}}(x) = N e^{-3\omega t/2} \int_0^\infty ds \sqrt{s} Z_k(se^{-\omega t}) e^{i\omega s \mathbf{n} \cdot \mathbf{x} - i\epsilon \ln s}. \quad (45)$$

Now using the scalar product (11) and the method of the Appendix B we can show that the normalization constant

$$N = \frac{1}{2} \sqrt{\frac{\omega}{2}} \frac{1}{(2\pi)^{3/2}}, \quad (46)$$

assures the desired orthonormalization relations

$$\langle f_{E,\mathbf{n}}, f_{E',\mathbf{n}'} \rangle = -\langle f_{E,\mathbf{n}}^*, f_{E',\mathbf{n}'}^* \rangle = \delta(E - E') \delta^2(\mathbf{n} - \mathbf{n}'), \quad (47)$$

$$\langle f_{E,\mathbf{n}}, f_{E',\mathbf{n}'}^* \rangle = 0, \quad (48)$$

and the completeness condition

$$i \int_0^\infty dE \int_{S^2} d\Omega_n \left\{ [f_{E,\mathbf{n}}(t, \mathbf{x})]^* \overleftrightarrow{\partial}_t f_{E,\mathbf{n}}(t, \mathbf{x}') \right\} = e^{-3\omega t} \delta^3(\mathbf{x} - \mathbf{x}'). \quad (49)$$

This means that the set of functions $\{f_{E,\mathbf{n}} | E \in \mathbb{R}^+, \mathbf{n} \in S^2\}$ constitutes the complete system of fundamental solutions of the *energy* basis of the NP.

The last step is to calculate the transition coefficients between the momentum and energy bases of the NP that read

$$\langle f_{\mathbf{p}}, f_{E,\mathbf{n}} \rangle = \langle f_{E,\mathbf{n}}, f_{\mathbf{p}} \rangle^* = \frac{p^{-3/2}}{\sqrt{2\pi\omega}} \delta^2(\mathbf{n} - \mathbf{n}_p) e^{-i\frac{E}{\omega} \ln \frac{p}{\omega}}, \quad (50)$$

where $\mathbf{n}_p = \mathbf{p}/p$. With their help we deduce the transformations

$$\begin{aligned} a(\mathbf{p}) &= \int_0^\infty dE \int_{S^2} d\Omega_n \langle f_{\mathbf{p}}, f_{E,\mathbf{n}} \rangle a(E, \mathbf{n}) \\ &= \frac{p^{-3/2}}{\sqrt{2\pi\omega}} \int_0^\infty dE e^{-i\frac{E}{\omega} \ln \frac{p}{\omega}} a(E, \mathbf{n}_p), \end{aligned} \quad (51)$$

$$\begin{aligned} a(E, \mathbf{n}) &= \int d^3p \langle f_{E,\mathbf{n}}, f_{\mathbf{p}} \rangle a(\mathbf{p}) \\ &= \frac{1}{\sqrt{2\pi\omega}} \int_0^\infty dp \sqrt{p} e^{i\frac{E}{\omega} \ln \frac{p}{\omega}} a(p \mathbf{n}), \end{aligned} \quad (52)$$

and similarly for the wave functions b . These relations will help us to perform the second quantization in canonical manner using both the bases defined above.

4 Quantization

The quantization can be done in canonical manner considering that the wave functions a and b of the fields (34) and (43) become field operators (such that $b^* \rightarrow b^\dagger$) [13]. We assume that the particle (a, a^\dagger) and antiparticle (b, b^\dagger) operators fulfill the standard commutation relations in the momentum basis, from which the non-vanishing ones are

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = [b(\mathbf{p}), b^\dagger(\mathbf{p}')] = \delta^3(\mathbf{p} - \mathbf{p}'). \quad (53)$$

Then, from Eq. (51) it results that the field operators of the energy basis satisfy

$$[a(E, \mathbf{n}), a^\dagger(E', \mathbf{n}')] = [b(E, \mathbf{n}), b^\dagger(E', \mathbf{n}')] = \delta(E - E') \delta^2(\mathbf{n} - \mathbf{n}'), \quad (54)$$

and

$$[a(\mathbf{p}), a^\dagger(E, \mathbf{n})] = \langle f_{\mathbf{p}}, f_{E,\mathbf{n}} \rangle, \quad (55)$$

while other commutators are vanishing. In this way the field ϕ is correctly quantized according to the *canonical* rule

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = e^{3\omega t} [\phi(t, \mathbf{x}), \partial_t \phi^\dagger(t, \mathbf{x}')] = i \delta^3(\mathbf{x} - \mathbf{x}'), \quad (56)$$

where $\pi = \sqrt{g} \partial_t \phi^\dagger$ is the momentum density derived from the action (3). All these operators act on the Fock space supposed to have an unique vacuum state $|0\rangle$ accomplishing

$$a(\mathbf{p})|0\rangle = b(\mathbf{p})|0\rangle = 0, \quad \langle 0|a^\dagger(\mathbf{p}) = \langle 0|b^\dagger(\mathbf{p}) = 0, \quad (57)$$

and similarly for the energy basis. The sectors with a given number of particles have to be constructed using the standard methods, obtaining thus the generalized bases of momentum or energy.

Furthermore, we have to calculate the one-particle operators corresponding to the conserved quantities (8). This can be achieved bearing in mind that for

any self-adjoint generator A of the scalar representation of the group $I(M)$ there exists a *conserved* one-particle operator of the quantum field theory which can be calculated simply as

$$\mathcal{A} =: \langle \phi, A\phi \rangle : \quad (58)$$

respecting the normal ordering of the operator products [13]. Hereby we recover the standard algebraic properties

$$[\mathcal{A}, \phi(x)] = -A\phi(x), \quad [\mathcal{A}, \mathcal{B}] =: \langle \phi, [A, B]\phi \rangle : \quad (59)$$

due to the canonical quantization adopted here. In other respects, the electric charge operator corresponding to the $U(1)$ internal symmetry (of Abelian gauge transformations $\phi \rightarrow e^{i\alpha I}\phi$) results from the Noether theorem to be $\mathcal{Q} =: \langle \phi, I\phi \rangle := : \langle \phi, \phi \rangle :$.

However, there are many other conserved operators which do not have corresponding differential operators at the level of quantum mechanics. The simplest examples are the operators of number of particles,

$$\mathcal{N}_{pa} = \int d^3p a^\dagger(\mathbf{p})a(\mathbf{p}) = \int_0^\infty dE \int_{S^2} d\Omega_n a^\dagger(E, \mathbf{n})a(E, \mathbf{n}), \quad (60)$$

and that of antiparticles, \mathcal{N}_{ap} (depending on b and b^\dagger), which give the charge operator $\mathcal{Q} = \mathcal{N}_{pa} - \mathcal{N}_{ap}$ and the operator of total number of particles, $\mathcal{N} = \mathcal{N}_{pa} + \mathcal{N}_{ap}$.

In what follows we focus on the conserved one-particle operators determining the momentum and energy bases. The diagonal operators of the momentum basis the are \mathcal{Q} and the components of momentum operator,

$$\mathcal{P}^i =: \langle \phi, \hat{P}^i \phi \rangle := \int d^3p p^i [a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p})] . \quad (61)$$

In other words, the momentum basis is determined by the set of commuting operators $\{\mathcal{Q}, \mathcal{P}^i\}$. The energy basis is formed by the common eigenvectors of the set of commuting operators $\{\mathcal{Q}, \mathcal{H}, \tilde{\mathcal{P}}^i\}$, i.e. the charge, energy and momentum direction operators. The energy operator can be easily calculated since the solutions (44) are eigenfunctions of the operator H . In this way we find

$$\mathcal{H} =: \langle \phi, H\phi \rangle := \int_0^\infty dE E \int_{S^2} d\Omega_n [a^\dagger(E, \mathbf{n})a(E, \mathbf{n}) + b^\dagger(E, \mathbf{n})b(E, \mathbf{n})] . \quad (62)$$

More interesting are the operators $\tilde{\mathcal{P}}^i$ of the momentum direction since they do not come from differential operators and, therefore, must be defined directly as

$$\tilde{\mathcal{P}}^i = \int_0^\infty dE \int_{S^2} d\Omega_n n^i [a^\dagger(E, \mathbf{n})a(E, \mathbf{n}) + b^\dagger(E, \mathbf{n})b(E, \mathbf{n})] . \quad (63)$$

The above operators which satisfy simple commutation relations,

$$[\mathcal{H}, \mathcal{P}^i] = i\omega \mathcal{P}^i, \quad [\mathcal{H}, \tilde{\mathcal{P}}^i] = 0, \quad [\mathcal{Q}, \mathcal{H}] = [\mathcal{Q}, \mathcal{P}^i] = [\mathcal{Q}, \tilde{\mathcal{P}}^i] = 0, \quad (64)$$

are enough for defining the bases considered here. However, there are more six conserved operators corresponding to the differential operators L_i and R^i , defined by Eqs. (15) and respectively (16), but their calculation according to the rule (58) requires special methods which will be discussed elsewhere.

Our approach offers the opportunity to provide closed expressions for the conserved one-particle operators in bases where these are not diagonal. For example we can calculate the energy operator in momentum basis either starting with the identity

$$(H f_{\mathbf{p}})(x) = -i\omega \left(p^i \partial_{p_i} + \frac{3}{2} \right) f_{\mathbf{p}}(x) \quad (65)$$

or using Eq. (52). The final result,

$$\mathcal{H} = \frac{i\omega}{2} \int d^3p p^i \left\{ \left[a^\dagger(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} a(\mathbf{p}) \right] + \left[b^\dagger(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} b(\mathbf{p}) \right] \right\}, \quad (66)$$

is similar with that obtained in Ref. [5] for the Dirac field on M .

We note that beside the above conserved operators we can introduce other one-particle operators extending the definition (58) to the non-conserved operators of our quantum mechanics. However, these operators will depend explicitly on time, their expressions being complicated and without an intuitive physical meaning.

In the quantum theory of fields it is important to study the Green functions related to the partial commutator functions (of positive or negative frequencies) defined as

$$D^{(\pm)}(x, x') = i[\phi^{(\pm)}(x), \phi^{(\pm)\dagger}(x')] \quad (67)$$

and the total one, $D = D^{(+)} + D^{(-)}$. These functions are solutions of the KG equation in both the sets of variables and obey $[D^{(\pm)}(x, x')]^* = D^{(\mp)}(x, x')$ such that D results to be a real function. This property suggests us to restrict ourselves to study only the functions of positive frequencies,

$$D^{(+)}(x, x') = i \int d^3p f_{\mathbf{p}}(x) f_{\mathbf{p}}(x')^* = i \int_0^\infty dE \int_{S^2} d\Omega_n f_{E,\mathbf{n}}(x) f_{E,\mathbf{n}}(x')^*, \quad (68)$$

resulted from Eqs. (34) and (43). Both these versions lead to the final expression

$$\begin{aligned} D^{(+)}(x, x') &= \frac{\pi}{4\omega} \frac{i}{(2\pi)^3} e^{-\frac{3}{2}\omega(t+t')} \\ &\times \int d^3p Z_k \left(\frac{p}{\omega} e^{-\omega t} \right) Z_k^* \left(\frac{p}{\omega} e^{-\omega t'} \right) e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \end{aligned} \quad (69)$$

from which we understand that $D^{(+)}(x, x') = D^{(+)}(t, t', \mathbf{x} - \mathbf{x}')$ and may deduce what happens at equal time. First we observe that for $t' = t$ the values of the function $D^{(+)}(t, t, \mathbf{x} - \mathbf{x}')$ are c-numbers which means that $D(t, t, \mathbf{x} - \mathbf{x}') = 0$. Moreover, from Eqs. (33) or (49) we find

$$(\partial_t - \partial_{t'}) D^{(+)}(t, t', \mathbf{x} - \mathbf{x}') \Big|_{t'=t} = e^{-3\omega t} \delta^3(\mathbf{x} - \mathbf{x}') \quad (70)$$

and similarly for $D^{(-)}$.

These functions help us to introduce the principal Green functions. In general, $G(x, x') = G(t, t', \mathbf{x} - \mathbf{x}')$ is a Green function of the KG equation if this obeys

$$(\partial_t^2 - e^{-2\omega t} \Delta_x + 3\omega \partial_t + m^2) G(x, x') = e^{-3\omega t} \delta^4(x - x'). \quad (71)$$

The properties of the commutator functions allow us to construct the Green function just as in the scalar theory on Minkowski spacetime. We assume that the retarded, D_R , and advanced, D_A , Green functions read

$$D_R(t, t', \mathbf{x} - \mathbf{x}') = \theta(t - t') D(t, t', \mathbf{x} - \mathbf{x}'), \quad (72)$$

$$D_A(t, t', \mathbf{x} - \mathbf{x}') = -\theta(t' - t) D(t, t', \mathbf{x} - \mathbf{x}'), \quad (73)$$

while the Feynman propagator,

$$\begin{aligned} D_F(t, t', \mathbf{x} - \mathbf{x}') &= i \langle 0 | T[\phi(x) \phi^\dagger(x')] | 0 \rangle \\ &= \theta(t - t') D^{(+)}(t, t', \mathbf{x} - \mathbf{x}') - \theta(t' - t) D^{(-)}(t, t', \mathbf{x} - \mathbf{x}'), \end{aligned} \quad (74)$$

is defined as a causal Green function. It is not difficult to verify that all these functions satisfy Eq. (71) if one uses the identity $\partial_t^2[\theta(t)f(t)] = \delta(t)\partial_t f(t)$, the artifice $\partial_t f(t - t') = \frac{1}{2}(\partial_t - \partial_{t'})f(t - t')$ and Eq. (70).

These Green functions have complicated structures such that it would be helpful to analyze their properties in momentum representation. However, this can not be achieved in NP where Eq. (71) does not have a convenient Fourier transformation because of its time dependence. Thus, the real challenge in further investigations will be the study of the Green functions in SP where this equation can be solved in momentum representation.

5 Concluding remarks

We presented here the complete quantum theory of the massive charged scalar field minimally coupled with the gravitation of the dS expanding universe, M . The main point of our approach is the new set of fundamental solutions of the energy basis, derived with the help of our SP. This new basis completes the framework of quantum theory, being crucial for understanding how can be measured the energy and momentum whose operators do not commute among themselves.

In our opinion there exists a *global* apparatus providing quantum observables globally defined on M , which have different forms in each local chart but global algebraic properties which do not depend on the chart we choose. In our case the global apparatus offer us a large collection of conserved operators corresponding to the dS isometries or some internal symmetries, determining quantum modes whose wave functions are: (I) solutions of the KG equation and (II) eigenfunctions of suitable systems of commuting conserved operators. In other words, the quantum states of the scalar field on M can be *prepared* only by the global apparatus since the KG equation is global.

Let us see how the global apparatus measures the energy and momentum of the one-particle state defined as

$$|\chi\rangle = \int d^3p \chi(\mathbf{p}) a^\dagger(\mathbf{p}) |0\rangle, \quad \chi(\mathbf{p}) = \rho(\mathbf{p}) e^{-i\vartheta(\mathbf{p})}, \quad (75)$$

where ρ and ϑ are real functions. The normalization condition

$$1 = \langle\chi|\chi\rangle = \int d^3p |\chi(\mathbf{p})|^2 = \int d^3p |\rho(\mathbf{p})|^2 \quad (76)$$

shows that the function ρ must be square integrable on \mathbb{R}_p^3 while ϑ remains an arbitrary function. Calculating the expectation values of the non-commuting operators \mathcal{P}^i and \mathcal{H} , we obtain first the usual formula of the momentum expectation value

$$\langle\chi|\mathcal{P}^i|\chi\rangle = \int d^3p p^i |\rho(\mathbf{p})|^2. \quad (77)$$

which is independent on ϑ . However, for the energy operator the result is rather surprising since from Eq. (66) we derive an expectation value

$$\langle\chi|\mathcal{H}|\chi\rangle = \omega \int d^3p [p^i \partial_{p_i} \vartheta(\mathbf{p})] |\rho(\mathbf{p})|^2 \quad (78)$$

which depends mainly on the phase ϑ . This means that we can prepare at anytime states with arbitrary desired expectation values of these observables. Thus, in the particular case of $\vartheta(\mathbf{p}) = \epsilon \ln(p)$ we obtain $\langle\chi|\mathcal{H}|\chi\rangle = \omega \epsilon = E$ indifferent on the form of ρ if this obeys the condition (76).

Beside the global apparatus, one may use other types of *local* apparatus [2] measuring quantum modes whose wave functions do not satisfy the condition (I). These form bases of the Fock space determined by sets of commuting operators which do not commute with the KG one. Consequently, the local apparatus can measure the parameters of some arbitrary quantum states in a given state prepared by the global apparatus but it is not able to prepare itself states of the KG field, remaining a mere detector. Moreover, when one detects quantum modes that are not genuine KG ones and, therefore, can not be correctly normalized, it appears the danger of creating the welling illusion even on M where the vacuum is stable.

We conclude that our approach seems to be coherent at the level of relativistic quantum mechanics where we give correct definitions of the principal observables and introduce the SP allowing us to find the energy basis. For this reason the second quantization in canonical manner leads to quantum free fields which can be manipulated simply as those of special relativity. These fields may be used now for calculating scattering processes on the dS expanding universe.

Appendix A: Special Bessel functions

Let us consider the Hankel functions $H_\nu^{(1,2)}(s)$ in the special case when $\nu = ik$ and define the functions Z_k such that

$$Z_k(s) = e^{-\pi k/2} H_{ik}^{(1)}(s), \quad Z_k^*(s) = e^{\pi k/2} H_{ik}^{(2)}(s). \quad (79)$$

Then, using the Wronskian W of the Bessel functions [14] we find that

$$Z_k^*(s) \overset{\leftrightarrow}{\partial}_s Z_k(s) = W[H_{ik}^{(2)}, H_{ik}^{(1)}] = \frac{4i}{\pi s}. \quad (80)$$

Appendix B: Normalization integrals

In spherical coordinates of the momentum space, $\mathbf{n} \sim (\theta_n, \phi_n)$, and the notation $\mathbf{q} = \omega s \mathbf{n}$, we have $d^3q = q^2 dq d\Omega_n = \omega^3 s^2 ds d\Omega_n$ with $d\Omega_n = d(\cos \theta_n) d\phi_n$. Moreover, we can write

$$\delta^3(\mathbf{q} - \mathbf{q}') = \frac{1}{q^2} \delta(q - q') \delta^2(\mathbf{n} - \mathbf{n}') = \frac{1}{\omega^3 s^2} \delta(s - s') \delta^2(\mathbf{n} - \mathbf{n}'), \quad (81)$$

where we denoted $\delta^2(\mathbf{n} - \mathbf{n}') = \delta(\cos \theta_n - \cos \theta'_n) \delta(\phi_n - \phi'_n)$.

The normalization integrals can be calculated either in SP using the scalar product (22) which leads to complicated calculations or simply in NP starting with the scalar product (11). According to Eqs. (45) and (81), this yields

$$\begin{aligned} \langle f_{E, \mathbf{n}}, f_{E', \mathbf{n}'} \rangle &= i \frac{N^2 (2\pi)^3}{\omega^3} \delta^2(\mathbf{n} - \mathbf{n}') \int_0^\infty \frac{ds}{s} e^{i(\epsilon - \epsilon') \ln s} \\ &\quad \times \left[Z_k^*(s e^{-\omega t}) \overset{\leftrightarrow}{\partial}_t Z_k(s e^{-\omega t}) \right]. \end{aligned} \quad (82)$$

Furthermore, using Eq. (80) and the identity

$$\frac{1}{2\pi\omega} \int_0^\infty \frac{ds}{s} e^{i(\epsilon - \epsilon') \ln s} = \delta(E - E'), \quad (83)$$

we find the value of the normalization constant (46).

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